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Genetic Algebras generated by $b-$bistochastic Quadratic Stochastic Operators: The Character and Associativity

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Abstract

In the present paper we consider a genetic algebra induced by $b-$bistochastic Quadratic Stochastic Operators (QSOs) which is called $b-$bistochastic genetic algebra. First, we characterize their nontrivial character function on $\mathbb{R}^n$. It turns out that, the given character function is not unique, hence full descriptions of such functions on $\mathbb{R}^1$ and $\mathbb{R}^2$ are established. Moreover, the defined algebra is commutative but not associative in general, hence, the associativity of $b-$bistochastic genetic algebras defined on $\mathbb{R}^1$ and $\mathbb{R}^2$ are described. In this work, the existence of non-trivial derivations on such algebras are given.

Keywords: non-associative algebra; genetic algebra; b-bistochastic; quadratic stochastic operators; non-linear operator.

1 Introduction

The theory of population genetics has been significantly contributed by the advancement of the study in genetic algebras such as baric, evolution, Bernstein, train, stochastic and etc [25, 27, 13]. A credit must be given to [4] for standardizing the language of abstract algebras to synchronize with genetic study. Original works on population genetics can be traced back to Bernstein's research where the evolution operators were investigated [2]. Later on, the quadratic stochastic operators (QSOs) generalize the concept of that evolution operators [13]. Dynamics of a QSO is closely related to the investigation of certain algebraic properties of the generated evolution algebra (see [3]).

For the sake of comprehension, let us consider QSOs in terms of biological interpretation. Let $I_n = \{1, 2, \dots, n\}$ be the n type of species (or traits) in a population and $\mathbf{x}^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$ represents the early state probability distribution of the species. By $P_{ij,k}$ we mean that the probability of an individual in the i^{th} species and j^{th} species to cross-fertilize (or interbreed) and produce an individual from k^{th} species (trait). For the given $\mathbf{x}^{(0)}$, we can find the probability distribution of the first generation, $\mathbf{x}^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)})$ by a total probability, i.e,

$$x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}, \ \ k \in \{1, \dots, n\}.$$

Repeating the similar procedures, one may get probability distribution of the second generation, $\mathbf{x}^{(2)}$ and so on. Hence, a QSO defines an evolution operator where it describes the probabilistic distribution of next generation if the initial probability state is given. Recent achievements and open problems in the theory of QSOs can be found in [20, 8] and the references therein.

Several studies have been made (see [6, 1, 23]) to investigate genetic algebras generated by QSOs which forms an algebraic structure on the vector space $\mathbb{R}^n$. Please refer also to [13, 21, 22] for the same studies. Certain relations between genetic and evolution algebras have been established in [16, 17]. We note that, the induced genetic algebras (via gametic, zygotic, or copular algebras) are generally commutative but non-associative, yet they are not necessarily Lie, Jordan, or alternative algebras. These algebraic properties have genetically meaning for modern uses, for instance, self-fertilization which can be found in [11]. Therefore, there are interactions between the purely mathematical structures and the corresponding genetic properties. We refer the readers to the article [27] for a comprehensive reference.

Based on the previous studies, it is apparent that investigation of QSOs in general is challenging (unlike in case of linear operators), therefore the researchers are likely to introduce classes of QSOs such as Volterra-QSOs [9], $b-$bistochastic QSOs [15], doubly QSOs [7], separable QSOs [26], ξ^s-QSOs [18], Geometric QSOs [12] and etc. In [6], the authors introduced genetic Volterra algebras and some of their algebraic properties were studied. Recently, connections between the evolution algebras and the associated dynamical system have been made in [24] for the case of Volterra QSOs. Motivated from those ideas, we are going to consider genetic algebras generated by $b-$bistochastic QSOs [15] which is simply called $b-$bistochastic genetic algebras. We point out that, the considered class of QSO's is a generalization to the concept of majorization that was popularized by Hardy, Littlewood and Polya [10]. Dynamical properties of $b-$bistochastic QSOs could be further referred to [15, 19].

This paper is organized as follows: Section 2 is devoted to preliminary definitions and notations throughout this paper. Descriptions of the character functions $b-$bistochastic genetic algebras are given in Section 3. In Section 4, we study the associativity of $b-$bistochastic genetic algebras. Last section is dedicated to the existence of non-trivial derivations $b-$bistochastic genetic algebras.

2 Preliminaries

This section presents necessary facts and notations throughout this paper. Let $I_n = \{1, \dots, n\}$. By $\{\mathbf{e}_i\}_{i \in I_n}$ we denote the standard basis in $\mathbb{R}^n$ i.e., $\mathbf{e}_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{in})$ where δ_{ij} are the Kronecker's Delta. Here and henceforth, we are going to consider the simplex:

$$S^{n-1} = \left\{ \mathbf{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \quad | x_i \geq 0, \quad \sum_{i=1}^{n} x_i = 1 \right\}. \tag{1}$$

Let us define an operator V that maps from S^{n-1} by:

$$V(\mathbf{x})_k = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j, \quad k \in I_n, \quad \mathbf{x} = (x_1, x_2, \dots, x_n) \in S^{n-1}, \tag{2}$$

where the heredity coefficients $P_{ij,k}$ satisfy

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{n} P_{ij,k} = 1, \quad \text{for any } \ i, j, k \in I_n. \tag{3}$$

One can check that V maps from the simplex S^{n-1} into itself. Such operator V is known as *Quadratic Stohastic Operator* (QSO). Suppose that $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ are arbitrary vectors in $\mathbb{R}^n$. By means of QSOs we introduce a multiplication rule (see [11]) on $\mathbb{R}^n$ as follows:

$$(\mathbf{x} \circ \mathbf{y})_k = \sum_{i,j=1}^{n} P_{ij,k} x_i y_j. \tag{4}$$

The pair $(\mathbb{R}^n, \circ)$ is called *genetic algebras*. Note that, the defined algebra is commutative, but not necessarily to be associative in general. Some algebraic properties of the defined algebras were studied in [11, 13, 27]. The authors in [6] define genetic algebras associated with a well-known class of QSOs, namely Volterra QSOs. Motivated from this idea, we are interested to induce the genetic algebras from $b-$bistochastic QSOs (see [15]). In what follows, the term $b-$**bistochastic genetic algebras** is referring to genetic algebras generated by $b-$bistochastic QSOs. Let us define functionals $\mathcal{U}_k : \mathbb{R}^n \to \mathbb{R}$ by:

$$\mathcal{U}_k(x_1, \dots, x_n) = \sum_{i=1}^{k} x_i \ \text{ where } \ k \in I_{n-1}. \tag{5}$$

Take any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in S^{n-1}$. We say that $\mathbf{x}$ is *b-ordered* by $\mathbf{y}$ ($\mathbf{x} \leq^b \mathbf{y}$) if and only if $\mathcal{U}_k(\mathbf{x}) \leq \mathcal{U}_k(\mathbf{y})$, for all $k \in I_{n-1}$. The introduced relation is indeed partial order i.e., it satisfies the following conditions:

(i) For any $\mathbf{x} \in S^{n-1}$ one has $\mathbf{x} \leq^b \mathbf{x}$,

(ii) If $\mathbf{x} \leq^b \mathbf{y}$ and $\mathbf{y} \leq^b \mathbf{x}$ then $\mathbf{x} = \mathbf{y}$,

(iii) If $\mathbf{x} \leq^b \mathbf{y}$, and $\mathbf{y} \leq^b \mathbf{z}$ then $\mathbf{x} \leq^b \mathbf{z}$.

Moreover, it has the following properties:

(i) One has that $\mathbf{x} \leq^b \mathbf{y}$ if and only if $\lambda\mathbf{x} \leq^b \lambda\mathbf{y}$ for any $\lambda > 0$

(ii) If $\mathbf{x} \leq^b \mathbf{y}$ and $\lambda \leq \mu$ then $\lambda\mathbf{x} \leq^b \mu\mathbf{y}$

Using this order, one can define the majorization [14]. First, recall that for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in S^{n-1}$, we write $\mathbf{x}_{[\downarrow]} = (x_{[1]}, x_{[2]}, \ldots, x_{[n]})$, where

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$$

is non-increasing rearrangement of $\mathbf{x}$. The point $\mathbf{x}_{[\downarrow]}$ is called *rearrangement* of $\mathbf{x}$ by nonincreasing. We say that $\mathbf{x}$ *is majorized by* $\mathbf{y}$ (or $\mathbf{y}$ $majorates$ $\mathbf{x}$) and denoted $\mathbf{x} \prec \mathbf{y}$ (or $\mathbf{y} \succ \mathbf{x}$) if $\mathbf{x}_{[\downarrow]} \leq^b \mathbf{y}_{[\downarrow]}$. Comprehensive information on this concept, results and applications could be found in [14]. One sees that $b-$order does not required the rearrangement of $\mathbf{x}$. Any operator V with $V(S^{n-1}) \subset S^{n-1}$ is called *stochastic*. In the same manner, if V satisfies $V(\mathbf{x}) \leq^b \mathbf{x}$ for all $\mathbf{x} \in S^{n-1}$, then it is called *$b-$bistochastic*.

The following theorems fully describe $b-$bistochastic QSOs defined on one and two dimensional simplices in terms of their heredity coefficients, respectively.

Theorem 2.1. [15] *Let V: $S^1 \longrightarrow S^1$ be a QSO, then V is a b-bistochastic if and only if*

$$P_{12,1} \leq \frac{1}{2}, \quad P_{22,1} = 0.$$

To describe two-dimensional QSOs, first, denote the heredity coefficients as follows:

$$\begin{array}{lll} P_{11,1} = A_1 & P_{13,1} = C_1 & P_{23,1} = E_1, \\ P_{11,2} = A_2 & P_{13,2} = C_2 & P_{23,2} = E_2, \\ P_{12,1} = B_1 & P_{22,1} = D_1 & P_{33,1} = F_1, \\ P_{12,2} = B_2 & P_{22,2} = D_2 & P_{33,2} = F_2. \end{array} \tag{6}$$

Theorem 2.2. [15] *Let V: $S^2 \longrightarrow S^2$ be a QSO, then V is a b-bistochastic if and only if*

(*i*) $F_1 = E_1 = D_1 = F_2 = 0;$

(*ii*) $B_1 \leq 1/2, C_1 \leq 1/2, E_2 \leq 1/2;$

(*iii*) $C_1 + C_2 \leq 1/2,$

and one of the following conditions satisfied :

(*I*) $\alpha \geq 0$;

(*II*) $\alpha < 0$ *and one of the following conditions satisfied:*

  (*1*) $\beta \leq 0$;

(2) $\beta \geq -2\alpha$;
(3) $\triangle \leq 0$.

where $\alpha = A_1 + A_2 + D_2 - 2B_1 - 2B_2,\ \beta = 2B_1 + 2B_2 - 2D_2,\ \gamma = D_2 - 1,\ \triangle = \beta^2 - 4\alpha\gamma.$

Necessary conditions for $b-$bistochastic QSOs in general setting are given as follows:

Theorem 2.1. [15] *Let* V *be a* $b-$*bistochastic QSO defined on* S^{n-1}, *then the following properties hold:*

$$(i)\ \sum_{m=1}^{k}\sum_{i,j=1}^{n} P_{ij,m} \leq kn;\quad k = \overline{1,n}$$

$$(ii)\ P_{ij,k} = 0 \text{ for } \forall\ i,j = \overline{k+1,n} \text{ where } k = \overline{1,n-1}$$

$$(iii)\ P_{nn,n} = 1$$

$$(iv)\ (a)\ V(\mathbf{x})_k = \sum_{l=1}^{k} P_{ll,k}x_l^2 + 2\sum_{l=1}^{k}\sum_{j=l+1}^{n} P_{lj,k}x_l x_j \ \text{ where }\ k = \overline{1,n-1}$$

$$(b)\ V(\mathbf{x})_n = x_n^2 + \sum_{l=1}^{n-1} P_{ll,n}x_l^2 + 2\sum_{l=1}^{n-1}\sum_{j=l+1}^{n} P_{lj,n}x_l x_j$$

$$(v)\ P_{lj,l} \leq \frac{1}{2}\ \text{ for all }\ l = \overline{1,n-1}\ \text{ where }\ j \geq l+1$$

$$(vi)\ P_{ll,l} + 2\sum_{j=l+1}^{n} P_{lj,l} + 2P_{ln,l}(l-n) \leq 1\ \text{ where }\ l = \overline{1,n-1}$$

Next, let us introduce the definition of Volterra QSOs. If the associated heredity coefficients $P_{ij,k}$ of a QSO V satisfy the following conditions:

$$P_{ij,k} = 0 \ \text{ for any } i,j \neq k, \tag{7}$$

then V is called a Volterra QSO. Biological interpretation of last statement is that the offspring k's typed is only possible to repeat the trait from its parents i.e., i's or j's typed. Using Theorem 2.1 and 2.2 one concludes the following remarks which explain the relationship between Volterra and $b-$bistochastic QSOs defined on one and two dimensional simplices, respectively.

Remark 2.2. *Let* $V : S^1 \to S^1$ *be a* $b-$*bistochastic QSOs, then* V *is Volterra QSOs if and only if* $P_{11,1} = 1$.

Remark 2.3. *Let* $V : S^2 \to S^2$ *be a* $b-$*bistochastic QSOs where the heredity coefficients is denoted by* (6) *, then* V *is Volterra if and only if* $A_1 = D_2 = 1$.

3 Character Function

In this section, we characterize the character functions of $b-$bistochastic genetic algebras. Let $\mathfrak{A}$ be a $b-$bistochastic genetic algebra, then a linear functional h from $\mathfrak{A}$ to $\mathbb{R}$ to be a character of $\mathfrak{A}$ if its satisfies $h(\mathbf{x} \circ \mathbf{y}) = h(\mathbf{x})h(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. One easily can see that if $h(\mathbf{x}) = 0$, then the functional h is a character of $\mathfrak{A}$. Such function is called trivial character, therefore it is interesting to describe all non-trivial case. We note that, h is a character function if and only if

$$h(\mathbf{e}_i \circ \mathbf{e}_j) = h(\mathbf{e}_i)h(\mathbf{e}_j) \text{ for any } i,j \in I_n. \tag{8}$$

Theorem 3.1. *For any n dimensional $b-$bistochastic genetic algebras $\mathfrak{A}$, then a functional given by*

$$h(\mathbf{x}) = \sum_{i=1}^{n} x_i. \tag{9}$$

is a character of $\mathfrak{A}$.

Proof. Let $h(\mathbf{x})$ be a linear functional given by (9), then by computing $\mathbf{e}_i \circ \mathbf{e}_j$ and taking into account the properties of heredity coefficients (see (3)), one finds that $h(\mathbf{e}_i \circ \mathbf{e}_j) = 1$. Thus, it is clear that $h(\mathbf{e}_i \circ \mathbf{e}_j) = h(\mathbf{e}_i)h(\mathbf{e}_j)$ for any $i, j \in \mathbb{N}$. Hence, $h(\mathbf{x})$ is a character of $\mathfrak{A}$. This ends the proof.

□

We stress that, non-trivial character functions of $b-$bistochastic genetic algebras may not be unique. Thus, a full description of their character functions defined on low dimensions will given in this section. For the sake of simplicity, the heredity coefficients, $P_{ij,k}$ of any QSO V defined on S^1 are denoted by:

$$a = P_{11,1}, \quad b = P_{12,1}, \quad c = P_{22,1}. \tag{10}$$

In case of $b-$bistochastic QSOs, $c = 0$.

Theorem 3.2. *Let $\mathfrak{A}$ be a $b-$bistochastic genetic algebra on $\mathbb{R}^2$. The character of $\mathfrak{A}$ takes the following forms:*

(i) $h(\mathbf{x}) = x_1 + x_2$.

(ii) $h(\mathbf{x}) = ax_1$ *if* $b = 0$.

for any $\mathbf{x} \in \mathfrak{A}$.

Proof. Statement (i) is obtained by means of Theorem 3.1. Next, let us prove for (ii) by supposing that $b = 0$. From $h(\mathbf{e}_2 \circ \mathbf{e}_2) = h(\mathbf{e}_2)h(\mathbf{e}_2)$, then one has $\gamma_2 - \gamma_2^2 = 0$ or equivalently $\gamma_2 \in \{0, 1\}$. If $\gamma_2 = 1$, then by $h(\mathbf{e}_1 \circ \mathbf{e}_2) = h(\mathbf{e}_1)h(\mathbf{e}_2)$ yields $\gamma_1 = 1$ i.e., (i). Therefore, take $\gamma_2 = 0$ and using $h(\mathbf{e}_1 \circ \mathbf{e}_1) = h(\mathbf{e}_1)h(\mathbf{e}_1)$ produces $\gamma_1(a - \gamma_1) = 0$. We conclude that $\gamma_1 = a$, hence proves this theorem. □

The rest of this section aims to describe the character functions of $b-$bistochastic genetic algebras defined on $\mathbb{R}^3$. Using (8), it follows that,

$$\begin{aligned} A_1\gamma_1 + A_2\gamma_2 + (1 - A_1 - A_2)\gamma_3 &= \gamma_1^2 && (11)\\ B_1\gamma_1 + B_2\gamma_2 + (1 - A_1 - A_2)\gamma_3 &= \gamma_1\gamma_2 && (12)\\ C_1\gamma_1 + C_2\gamma_2 + (1 - C_1 - C_2)\gamma_3 &= \gamma_1\gamma_3 && (13)\\ D_2\gamma_2 + (1 - D_2)\gamma_3 &= \gamma_2^2 && (14)\\ E_2\gamma_2 + (1 - E_2)\gamma_3 &= \gamma_2\gamma_3 && (15)\\ \gamma_3 &= \gamma_3^2. && (16) \end{aligned}$$

Now, let us formulate main result in this section.

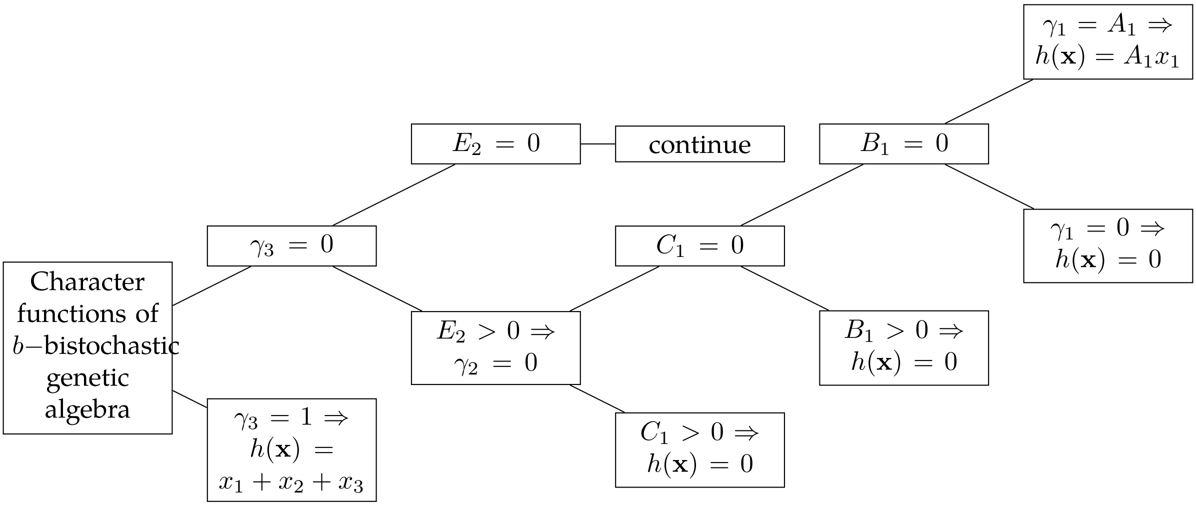

Figure 1: Summary of all possible cases to describe the character function on $\mathbb{R}^3$

Theorem 3.3. *Let $\mathfrak{A}$ be a $b-$bistochastic genetic algebra defined on $\mathbb{R}^3$. The character function of $\mathfrak{A}$ has non-trivial forms as follows:*

(*i*) $h(\mathbf{x}) = \gamma_1x_1 + \gamma_2x_2$ *where*

$$\gamma_1 = \frac{-C_2D_2}{C_1}, \ \ \gamma_2 = D_2$$

and $B_2C_1 + C_2D_2 - B_1C_2 = 0, \ C_1^2A_2 - A_1C_1C_2 - C_2^2D_2 = 0.$

(*ii*) $h(\mathbf{x}) = \gamma_1x_1 + \gamma_2x_2$ *where*

$$\gamma_1 = \frac{B_2D_2}{D_2 - B_1}, \ \ \gamma_2 = D_2$$

and $C_2 = 0, \ D_2 \neq B_2, \ A_1B_2D_2(D_2 - B_1) + A_2D_2(D_2 - B_1)^2 - B_2^2D_2^2 = 0.$

(*iii*) $h(\mathbf{x}) = \gamma_1x_1 + \gamma_2x_2$ *where*

$$\gamma_1 = \frac{A_1 \pm \sqrt{\triangle}}{2}, \ \ \gamma_2 = D_2,$$

and, $B_2 = 0, \ \triangle = A_1^2 - 4A_2B_1 \geq 0.$

(*iv*) $h(\mathbf{x}) = A_1x_1$ *for* $A_1 \neq 0.$

(*v*) $h(\mathbf{x}) = x_1 + x_2 + x_3.$

Proof. Assume that the linear functional, $h(\mathbf{x}) = \gamma_1x_1 + \gamma_2x_2 + \gamma_3x_3$ is a character function of $\mathfrak{A}$, then we shall have all equations (11) to (16). From (16) yields $\gamma_3 = 0$ or $\gamma_3 = 1$ (see Figure 1). If $\gamma_3 = 1$, then by the same method provided in the proof of Theorem 3.1 we get (v). Thus, let us assume that $\gamma_3 = 0$. Our interest is to find the value for γ_1 and γ_2. Hence, (16) becomes $E_2\gamma_2 = 0$, so again it is either $E_2 = 0$ or $E_2 > 0$ (see Figure 1).

Case 1: Let $E_2 > 0$. From (15) one infers that $\gamma_2 = 0$. Consequently, if either $B_1 > 0$ or $C_1 > 0$, we get $\gamma_1 = 0$ (see (12) and (13)), hence a trivial character. Therefore, we let $B_1 = C_1 = 0$. From (11) one gets $\gamma_1 = A_1$ or else $\gamma_1 = 0$ which is again a trivial case. Without the loss of generality we may assume that $A_1 \neq 0$ which shows (iv).

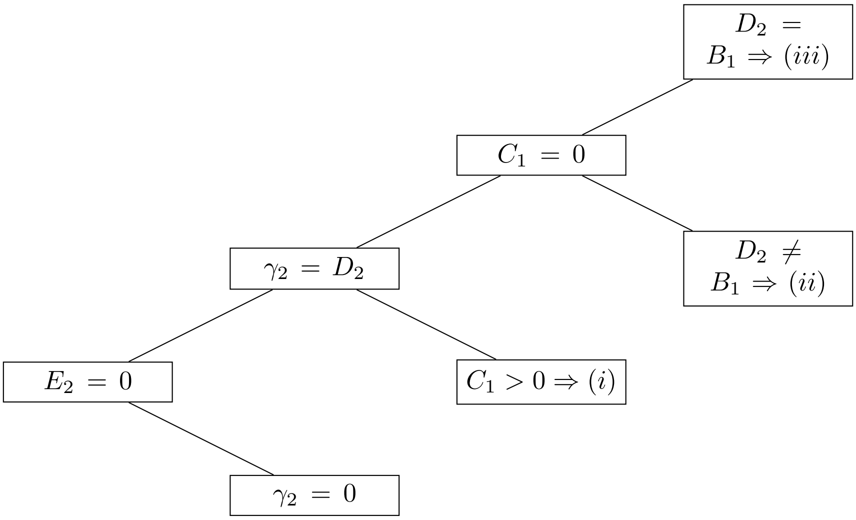

Figure 2: A continuation of Figure 1

Case 2: Let us consider $E_2 = 0$. Taking into account (14) we obtain

$$\gamma_2(D_2 - \gamma_2) = 0,$$

which implies either $\gamma_2 = 0$ or $\gamma_2 = D_2$ (see Figure 2). Suppose that $\gamma_2 = 0$, then it repeats again Case 1, thus $\gamma_2 = D_2$ (without the lost of generality we may assume that $D_2 \neq 0$) and if $C_1 > 0$, then from (13) yields $C_1\gamma_1 + C_2D_2 = 0$. Consequently, one gets $\gamma_1 = \dfrac{-C_2D_2}{C_1}$. Substituting the value of γ_1 and γ_2 into (11) and (12) we find that $C_1^2A_2 - A_1C_1C_2 - C_2^2D_2 = 0$ and $B_2C_1 + C_2D_2 - B_1C_2 = 0$, respectively. This proves (i).

(ii). From $\gamma_2 = D_2$ (see Figure 2) we suppose that $C_1 = 0$, then by (13) one has $C_2D_2 = 0$ which implies that $C_2 = 0$. Moreover (12) reduces to

$$B_1\gamma_1 + B_2D_2 = \gamma_1D_2,$$

which yields $\gamma_1 = \dfrac{B_2D_2}{D_2 - B_1}$ only if $D_2 \neq B_1$. Taking into account (11) one has $A_1B_2D_2(D_2 - B_1) + A_2D_2(D_2 - B_1)^2 - B_2^2D_2^2 = 0$, thus shows (ii).

(iii) As a continuation from (ii), we suppose that $D_2 = B_1$. Then (12) gives $B_1B_2 = 0$ which means $B_2 = 0$. By solving the value γ_1 from (11) one gets

$$\gamma_1 = \frac{A_1 \pm \sqrt{\triangle}}{2},$$

where $\triangle = A_1^2 - 4A_2B_1 \geq 0$ i.e., (iii). The proof is completed. $\square$

4 Section 4: Associativity of $b-$bistochastic Genetic Algebras

The objective in this section is to provide a complete description of associativity $b-$bistochastic genetic algebras defined on $\mathbb{R}^2$ and $\mathbb{R}^3$. Based on the definition of associativity, one may show that a genetic algebra $\mathfrak{A}$ is associative if and only if

$$(\mathbf{e}_i \circ \mathbf{e}_j) \circ \mathbf{e}_k = \mathbf{e}_i \circ (\mathbf{e}_j \circ \mathbf{e}_k) \text{ satisfy for all } i, j, k \in I_n. \tag{17}$$

Let us consider $b-$bistochastic genetic algebras $\mathfrak{A}$ on $\mathbb{R}^2$ where the associated heredity coefficients is denoted by (10).

From Theorem 2.1 one has the following multiplication table:

$\circ$	e_1	e_2
e_1	$(a\,;1-a)$	$(b\,;1-b)$
e_2	$(b\,;1-b)$	$(0\,;1)$

Table 1: Multiplication Table $(\mathbb{R}^2, \circ)$

Next theorem describes associativity of $\mathfrak{A}$ on $\mathbb{R}^2$.

Theorem 4.1. *Two dimensional $b-$bistochastic genetic algebras are associative if and only if $b = 0$.*

Proof. Let us assume that $\mathfrak{A}$ is associative, then taking into account Table 1, we have

$$(\mathbf{e}_1 \circ \mathbf{e}_1) \circ \mathbf{e}_2 = \mathbf{e}_1 \circ (\mathbf{e}_1 \circ \mathbf{e}_2) \Rightarrow ab\mathbf{e}_1 + (1 - ab)\mathbf{e}_2 = (ab + b(1-b)\mathbf{e}_1 + ((1-a)b + (1-b)^2)\mathbf{e}_2.$$

Solving the equality above, we obtain the following quadratic equation:

$$b^2 - b = 0,$$

which has roots $b = 0$ and $b = 1$. Thanks to Theorem 2.1, we infer that $b = 0$. Vice verse, suppose that $b = 0$, we are going to show $(\mathbf{e}_i \circ \mathbf{e}_j) \circ \mathbf{e}_k = \mathbf{e}_i \circ (\mathbf{e}_j \circ \mathbf{e}_k)$ are satisfied for all $i, j, k \in I_2$. Observe that, for $i = j = k = 1$ and $i = j = 1, k = 2$ then the statements are obvious. In case of $i = k = 1, j = 2$ the associativity is satisfied since,

$$(\mathbf{e}_1 \circ \mathbf{e}_2) \circ \mathbf{e}_1 = \mathbf{e}_2 \circ \mathbf{e}_1 = \mathbf{e}_1 \circ \mathbf{e}_2 = \mathbf{e}_1 \circ (\mathbf{e}_2 \circ \mathbf{e}_1).$$

Others can be shown similarly. This finishes the proof. □

Full description of two-dimensional genetic algebra could be found in [5]. Unfortunately, the proof was not given.

Furthermore, let us consider three dimensional b-bistochastic genetic algebras where the associated heredity coefficients are given by (6). Based on Theorem 2.2 one gets the following multiplication table:

$\circ$	e_1	e_2	e_3
e_1	$(A_1\,; A_2\,; 1 - A_1 - A_2)$	$(B_1\,; B_2; 1 - B_1 - B_2)$	$(C_1\,; C_2\,; 1 - C_1 - C_2)$
e_2	$(B_1\,; B_2\,; 1 - B_1 - B_2)$	$(0\,; D_2\,; 1 - D_2)$	$(0\,; E_2\,; 1 - E_2)$
e_3	$(C_1\,; C_2\,; 1 - C_1 - C_2)$	$(0\,; E_2\,; 1 - E_2)$	$(0\,; 0\,; 1)$

Table 2: Multiplication Table $(\mathbb{R}^3, \circ)$

Theorem 4.1. *Three dimensional $b-$bistochastic genetic algebras are associative if and only if $C_1 = C_2 = E_2 = 0$ and either (i) or (ii) is satisfied. Here (i) and (ii) are given by,*

(i) *$B_2 = 0$ and either*
  (a) *$A_2 = 0$ and $B_1 = 0$ or*
  (b) *$B_1 = D_2$.*
(ii) *$B_2 = A_1 > 0$ and $B_1 = D_2 = 0$.*

Proof. Let us prove "*if*" part. Associativity condition gives:

$$(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k) \text{ for all } i, j, k \in I_3. \tag{18}$$

By considering $(\mathbf{e}_1 \circ \mathbf{e}_3) \circ \mathbf{e}_3 = \mathbf{e}_1 \circ (\mathbf{e}_3 \circ \mathbf{e}_3)$ and $(\mathbf{e}_2 \circ \mathbf{e}_3) \circ \mathbf{e}_3 = \mathbf{e}_2 \circ (\mathbf{e}_3 \circ \mathbf{e}_3)$ one has

$$C_1(C_1 - 1) = 0 \text{ and } \ E_2(E_2 - 1) = 0,$$

respectively. The last statement and Theorem 2.2 imply $C_1 = 0$ and $E_2 = 0$. This result togehter with $(\mathbf{e}_1 \circ \mathbf{e}_3) \circ \mathbf{e}_3 = \mathbf{e}_1 \circ (\mathbf{e}_3 \circ \mathbf{e}_3)$ produce $C_2 = 0$.

Therefore, (18) reduces to

$$\begin{aligned}
B_1B_2 &= 0 && (19)\\
B_2(A_1 - B_2) + A_2(D_2 - B_1) &= 0 && (20)\\
A_2D_2 + B_1(-A_2 - B_2) + B_2(A_1 - B_2) &= 0 && (21)\\
B_1(B_1 - D_2) &= 0 && (22)\\
B_1(D_2 - B_1 - B_2) &= 0 && (23)\\
B_1(D_2 - B_1) &= 0 && (24)\\
B_1(B_1 + B_2 - D_2) &= 0 && (25)\\
A_2(D_2 - B_1) + B_2(A_1 - B_1 - B_2) &= 0 && (26)\\
B_2D_2 &= 0. && (27)
\end{aligned}$$

Taking into account (19), one can check that (20), (21) and (26) are the same. Moreover, (22) and (24) are clearly equivalent. Keeping in mind (19) and (22), the equations (23) and (25) are obviously satisfied. These simplifications yield the following a system of equations:

$$\begin{cases} B_1B_2 &= 0\\ B_2(A_1 - B_2) + A_2(D_2 - B_1) &= 0\\ B_1(B_1 - D_2) &= 0\\ B_2D_2 &= 0. \end{cases} \tag{28}$$

From now on, we divide into two cases i.e., *Case 1:* $B_2 = 0$ and *Case 2:* $B_2 > 0$.

Case 1: Let $B_2 = 0$. Then (28) becomes

$$\begin{cases} A_2(D_2 - B_1) &= 0\\ B_1(B_1 - D_2) &= 0, \end{cases} \tag{29}$$

which is evidently satisfied if either $A_2 = B_1 = 0$ or $B_1 = D_2$. This proves (i).

Case 2: Let $B_2 > 0$. Then $B_1 = D_2 = 0$, therefore (28) reduces to

$$B_2(A_1 - B_2) = 0,$$

that implies $A_1 = B_2$. Hence, proves (ii). Next, "*only if*" part is obvious due to the calculations above. This ends the proof. ☐

We stress that, the description of the associativity in general setting is tricky since the necessary and sufficient conditions defined on any finite dimension is not known in the literature.

5 Derivation of $b-$bistochastic Genetic Algebras

In this section, we show that the existence of non-trivial derivation of considered algebras.

Let $(\mathfrak{A}, \circ)$ be a genetic algebra. Recall that, a linear mapping $D : \mathfrak{A} \to \mathfrak{A}$ is called *derivation* of $\mathfrak{A}$ if the mapping satisfies

$$D(\mathbf{x} \circ \mathbf{y}) = D(\mathbf{x}) \circ \mathbf{y} + \mathbf{x} \circ D(\mathbf{y}), \tag{30}$$

for any element $\mathbf{x}, \mathbf{y} \in \mathfrak{A}$. By means of linearity one has D is derivative if and only if

$$D(\mathbf{e}_i \circ \mathbf{e}_j) = D(\mathbf{e}_i) \circ \mathbf{e}_j + \mathbf{e}_i \circ D(\mathbf{e}_j), \tag{31}$$

for all $i, j \in I_n$.

The following theorem describes the derivation of two dimensional $b-$bistochastic genetic algebras.

Theorem 5.1. *Let $\mathfrak{A}$ be two dimensional b-bistochastic genetic algebras. Then the following statements hold:*

- (i) *If $b \neq {}^1\!/_2$ and $b \neq {}^a\!/_2$, then all derivatives are trivial.*
- (ii) *If $b = {}^1\!/_2$ and $a \neq 1$, then all derivatives are trivial.*
- (iii) *If $b = {}^a\!/_2$ for any $a \in [0, 1)$, then the derivative has the following form:*

$$D(\mathbf{x}) = x_1 t(\mathbf{e}_2 - \mathbf{e}_1) \ \textit{for any} \ \ t \in \mathbb{R}.$$

- (iv) *If $b = {}^1\!/_2$ and $a = 1$, then the derivative has the following form:*

$$D(\mathbf{x}) = \mathbf{e}_1(sx_2 - tx_1) + \mathbf{e}_2(tx_1 - sx_2) \ \textit{for any} \ \ s, t \in \mathbb{R}.$$

Proof. Let D be a linear operator given by

$$D(\mathbf{e}_i) = \sum_{j=1}^{n} d_{ij} \mathbf{e}_j, \tag{32}$$

is a derivation of $\mathfrak{A}$. Then, (31) leads to:

$$\begin{aligned}
&ad_{11} + 2bd_{12} - (1-a)d_{21} = 0 && (33)\\
&2(1-a)d_{11} + (2(1-b)-a)d_{12} - (1-a)d_{22} = 0 && (34)\\
&(2b-1)d_{21} = 0 && (35)\\
&2(1-b)d_{21} + d_{22} = 0 && (36)\\
&(1-b-a)d_{21} - bd_{22} = 0 &&\\
&(b-1)d_{12} - (1-b)d_{11} - (1-a)d_{21} = 0 &&
\end{aligned}$$

Taking into account Theorem 2.1 and (35), we can consider two cases $b < \frac{1}{2}$ and $b = \frac{1}{2}$.

CASE I: Let $b < \frac{1}{2}$, then from (35) one gets $d_{21} = 0$. This fact together with (36) yields $d_{22} = 0$. Further, substituting (33) into (34) reduces the last system of equations to:

$$\begin{aligned}
ad_{11} + 2bd_{12} &= 0. && (37)\\
(2-a)(d_{11} + d_{12}) &= 0. && (38)\\
(1-b)(d_{12} + d_{11}) &= 0. && (39)
\end{aligned}$$

Since $a \neq 2$, then (38) implies $d_{12} = -d_{11}$. The last equality together with (37) produce $(a - 2b)d_{11} = 0$. Therefore, we divide into two parts.

Part 1: Suppose $b \neq \frac{a}{2}$, then $d_{11} = d_{12} = d_{21} = d_{22} = 0$ i.e., the derivative D is trivial, hence proves (i).

Part 2: Let $b = \frac{a}{2}$. It should be clear that d_{11} becomes a free variable i.e., $d_{11} = t \in \mathbb{R}$. Thus,

$$D(\mathbf{x}) \quad = \quad x_1 t(\mathbf{e}_2 - \mathbf{e}_1),$$

which yields (iii).

Case II: In this case, we let $b = \frac{1}{2}$. Using (31) one gets the following system of equations:

$$\begin{aligned}
ad_{11} + d_{12} - (1-a)d_{21} &= 0\\
2(1-a)d_{11} + (1-a)d_{12} - (1-a)d_{22} &= 0\\
d_{21} + d_{22} &= 0 \qquad (40)\\
\frac{1}{2}(d_{21} - d_{22}) - ad_{21} &= 0\\
\frac{1}{2}(d_{11} + d_{12}) + (1-a)d_{21} &= 0.
\end{aligned}$$

Due to (40) the system becomes:

$$\begin{aligned}
ad_{11} + d_{12} - (1-a)d_{21} &= 0 && (41)\\
(1-a)(2d_{11} + d_{12} + d_{21}) &= 0 && (42)\\
(1-a)d_{21} &= 0 && (43)\\
\frac{1}{2}(d_{11} + d_{12}) + (1-a)d_{21} &= 0 && (44)
\end{aligned}$$

Referring to the equation (43), again we divide into two parts.

Part 1: Let $a \neq 1$, thus $d_{21} = 0$. From (44), one has $d_{11} = -d_{12}$ and together with (41) produce $d_{11} = d_{12} = 0$. Therefore, the derivation D is trivial i.e., (ii).

Part 2: Let $a = 1$. Taking into account (41), we get $d_{11} = -d_{12}$. Letting $d_{12} = t$ and $d_{21} = s$ then the derivation takes the following form:

$$D(X) \quad = \quad \mathbf{e}_1(sx_2 - tx_1) + \mathbf{e}_2(tx_1 - sx_2)$$

which proves(iv).

□

We stress that in [6], it was given a full description of derivation genetic Volterra algebra on three dimension. Namely,

Theorem 5.1. [6] *Let $\mathfrak{A}$ be a three dimensional genetic Volterra algebra. The algebra has a non-trivial derivation if and only if there exists i, j, k with $i, j, k \in I_3$ such that $P_{ij,i} = \frac{1}{2}$, $P_{ik,i} = P_{jk,j}$.*

Recall Remark 2.2, there are similarities between $b-$bistochastic and Volterra QSOs defined on S^2, hence it is interesting to study the derivation genetic algebras generated by purely $b-$bistochastic QSOs. We say that purely $b-$bistochastic QSOs associated with Class (i) if $A_2 > 0$, Class (ii) if $B_1 + B_2 < 1$, Class (iii) if $C_2 > 0$, Class (iv) if $D_2 < 1$. Based on the definition of Volterra, these classes are not Volterra, for instance Class (i) is not Volterra since $A_2 = P_{11,2} > 0$ (see (7)).

Here and henceforth, by purely $b-$bistochastic QSOs, we are referring the operator which is associated with Class (iii) only. One observes that, from the assumption $C_2 > 0$ and (iii) in Theorem 2.2 imply $C_1 < \frac{1}{2}$. Let us provide an auxiliary result as follows:

Proposition 5.2. *Let $\mathfrak{A}$ be a b-bistochastic genetic algebra. If D is a derivation, then*

$$\sum_{j=1}^{3} d_{ij} = 0 \textit{ for any } i \in \{1, 2, 3\}.$$

Proof. Let us assume that D is derivation for $\mathfrak{A}$. Using Table 2 one has

$$\begin{aligned} D(\mathbf{e}_3) &= 2D(\mathbf{e}_3) \circ \mathbf{e}_3 \\ &= 2\left(\sum_{j=1}^{3} d_{3j}\mathbf{e}_j\right) \circ \mathbf{e}_3 \\ &= 2\sum_{j=1}^{3} d_{3j}\left(\mathbf{e}_j \circ \mathbf{e}_3\right) \\ &= 2C_1 d_{31}\mathbf{e}_1 + 2(d_{31}C_2 + d_{32}E_2)\mathbf{e}_2 + 2(d_{31}(1 - C_1 - C_2) + d_{32}(1 - E_2) + d_{33})\mathbf{e}_3 \end{aligned}$$

Comparing both hand sides, one obtains the following system of equations

$$d_{31} - 2d_{31}C_1 \quad = \quad 0; \tag{45}$$

$$d_{32} - 2\,d_{31}\,C_2 - 2\,d_{32}\,E_2 \quad = \quad 0; \tag{46}$$

$$-d_{33} - 2\,d_{31} + 2\,d_{31}\,C_1 + 2\,d_{31}\,C_2 - 2\,d_{32} + 2\,d_{32}\,E_2 \quad = \quad 0. \tag{47}$$

One easily can check that the sum of (45), (46) and (47) produce

$$d_{31} + d_{32} + d_{33} = 0. \tag{48}$$

Further, from $D(\mathbf{e}_2 \circ \mathbf{e}_2) = 2D(\mathbf{e}_2) \circ \mathbf{e}_2$ we have the following system of equations

$$\begin{aligned}
d_{21}\,D_2 + d_{31} - d_{31}\,D_2 - 2\,d_{21}\,B_1 - 2\,d_{23}\,C_1 &= 0 && (49)\\
-d_{22}\,D_2 + d_{32} - d_{32}\,D_2 - 2\,B_2\,d_{21} - 2\,d_{23}\,C2 &= 0 && (50)\\
D_2\,d_{23} + d_{33} - d_{33}\,D_2 - 2\,d_{21} + 2\,d_{21}\,B_1 + 2\,B_2\,d_{21} - 2\,d_{22} + & && (51)\\
2\,D_2\,d_{22} - 2\,d_{23} + 2\,d_{23}\,C_1 + 2\,d_{23}\,C_2 &= 0
\end{aligned}$$

Adding up (49) to (51) and taking into account (48) give

$$(d_{21} + d_{22} + d_{23})(D_2 - 2) = 0$$

Since $D_2 \le 1$, we infer

$$d_{21} + d_{22} + d_{23} = 0. \tag{52}$$

Using similar argument, one can consider the equalities $D(\mathbf{e}_1 \circ \mathbf{e}_2) = 2D(\mathbf{e}_1) \circ \mathbf{e}_1$, (48) and (52) to show that

$$d_{11} + d_{12} + d_{13} = 0. \tag{53}$$

The obtained equations (48), (52) and (53) prove the proposition. □

Theorem 5.3. *Let $\mathfrak{A}$ be a purely b-bistochastic genetic algebra. If $E_2 < \frac{1}{2}$ and $C_1 > 0$, then the derivation is trivial.*

Proof. Suppose that $\mathfrak{A}$ be a purely b-bistochastic genetic algebra with $E_2 < \frac{1}{2}$ and $C_1 > 0$. Let D be a derivation given by (32). From (45) and early assumption, we conclude that $d_{31} = 0$. This result together with (46) yield $d_{32} = 0$. Using Proposition 5.2 we get $d_{33} = 0$. Substituting the last results into $D(\mathbf{e}_3 \circ \mathbf{e}_2) = D(\mathbf{e}_3) \circ \mathbf{e}_2 + \mathbf{e}_3 \circ D_{\mathbf{e}_2}$ and it produces a system of equations as follows:

$$\begin{aligned}
d_{21}E_2 - d_{21}C_1 &= 0\\
-d_{21}C_2 &= 0\\
d_{23}E_2 - d_{21} + d_{21}C_1 + d_{21}C_2 - d_{22} + d_{22}E_2 - d_{23} &= 0
\end{aligned} \tag{54}$$

Due to $C_2 > 0$ and (54), one concludes $d_{21} = 0$. Moreover, from (49), one has $-2d_{23}C_1 = 0$. Since $C_1 > 0$, thus $d_{23} = 0$. Using Proposition 5.2 we get $d_{22} = 0$. By computing $D(\mathbf{e}_2 \circ \mathbf{e}_3) = D(\mathbf{e}_2) \circ \mathbf{e}_3 + \mathbf{e}_2 \circ D(\mathbf{e}_3)$ one has $d_{11} = d_{12} = d_{13} = 0$, hence a trivial derivation. This ends the proof. □

A natural question may arise, is there any non-trivial derivation for purely $b-$bistochastic genetic algebras. This question is answered in next theorem.

Theorem 5.4. *If $A_1 = 2B_1, A_2 = B_2 - B_1, D_2 = B_1 + B_2$ and $E_2 = C_2$, then there exists non-trivial derivation of $b-$bistochastic genetic algebras.*

Proof. From Theorem 2.2, one easily can check that $\alpha = 0$ i.e., the selected heredity coefficients $P_{ij,k}$ is indeed $b-$bistochastic QSOs. Let us define the functional D in the form:

$$D(\mathbf{e}_1) = t(\mathbf{e}_1 - \mathbf{e}_2), D(\mathbf{e}_2) = 0, D(\mathbf{e}_3) = 0 \quad \text{for any } t \in \mathbb{R}.$$

To prove D is a derivation, we are going to use Lemma 31. By mean of Table 2 one has

$$\begin{aligned} D(\mathbf{e}_1 \circ \mathbf{e}_1) &= D(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + (1 - A_1 - A_3)\mathbf{e}_3) \\ &= A_1 D(\mathbf{e}_1) + A_2 D(\mathbf{e}_2) + (1 - A_1 - A_2) D(\mathbf{e}_3) \\ &= 2tB_1(\mathbf{e}_1 - \mathbf{e}_2). \end{aligned}$$

On the other hands,

$$\begin{aligned} 2D(\mathbf{e}_1) \circ \mathbf{e}_1 &= 2t(\mathbf{e}_1 - \mathbf{e}_2) \circ \mathbf{e}_1 \\ &= 2t\mathbf{e}_1 \circ \mathbf{e}_1 - 2t\mathbf{e}_2 \circ \mathbf{e}_1 \\ &= 2t(A_1\mathbf{e}_1 + A_2\mathbf{e}_2 + (1 - A_1 - A_2)\mathbf{e}_3) - 2t(B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + (1 - B_1 - B_2)\mathbf{e}_3) \\ &= 2t(2B_1\mathbf{e}_1 + (B_2 - B_1)\mathbf{e}_2 + (1 - 2B_1 - B_2 + B_1)\mathbf{e}_3) \\ &\quad -2t(B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + (1 - B_1 - B_2)\mathbf{e}_3) \\ &= 2tB_1(\mathbf{e}_1 - \mathbf{e}_2) \end{aligned}$$

i.e., $D(\mathbf{e}_1 \circ \mathbf{e}_1) = 2D(\mathbf{e}_1) \circ \mathbf{e}_1$. Using the similar method one can prove that $D(\mathbf{e}_i \circ \mathbf{e}_j) = D(\mathbf{e}_i) \circ \mathbf{e}_j + \mathbf{e}_i \circ D(\mathbf{e}_j)$ is satisfied for all $i, j = \{1, 2, 3\}$. Thus D is a derivation for $\mathfrak{A}$, hence completes the proof. ☐

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